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ON THE EQUIVALENCE OF A SET OF MUTUALLY ORTHOGONAL LATIN SQUARES WITH OTHER COMBINATORIAL SYSTEMS [1]

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ABSTRACT

In this expository paper we have densely summarized some of the results obtained by author and at least fourteen others in order to demonstrate the importance of the theory of mutually orthogonal latin squares. We have shown that fourteen well-known and important combinatorial systems with certain parameters are actually equivalent to a set of mutually orthogonal latin squares. A schematic representation of these equivalences have been demonstrated in four wheels which we have called them "Fundamental Wheels of Combinatorial Mathematics".

INTRODUCTION: The theory of mutually orthogonal latin squares owes its importance to the fact that many well-known combinatorial systems are actually equivalent to a set of mutually orthogonal latin squares; viz, finite projective plane, finite Euclidean plane, net, BIB, PBIB, orthogonal arrays, a set of mutually orthogonal matrices, error correcting codes, strongly regular graphs, complete graphs, a balanced set of ℓ-restrictional lattice designs, difference sets, Hadamard matrices, and an arrangement of non attacking rooks on hyperdimensional chess board. These combinatorial systems are unquestionably potent and effective in all branches of combinatorial mathematics, and in particular, in the construction of experimental designs. Therefore, a statement that the theory of mutually orthogonal latin squares is perhaps the most important theory <u>in the field of experimental designs</u> is not in the least exaggerated as far as this author is concerned.

Our purpose in this paper is to demonstrate the relation of a set of mutually orthogonal latin squares with the above mentioned combinatorial systems. We shall present the essence of the known results available only in scattered literature in one theorem which we consider to be a "fundamental theorem of combinatorial mathematics". For the definitions of these combinatorial systems and the proof of the forthcoming theorem see the list of references given at the end of this paper.

I. NOTATIONS: For the sake of conciseness we introduce the following notations:

0) $O(n,t)$ denotes a set of t mutually orthogonal latin squares of order n.

1) $MOM(n,t)$ denotes a set of t mutually orthogonal $n \times n$ matrices.

2) $OA(n,t)$ denotes a set of orthogonal arrays of size n^2, depth t, n levels and strength 2.

3) $Net(n,t)$ denotes a net of order n and degree t.

4) $Code(n,r,t;m)$ denotes a set of n code words each of length r such that any two code words are at least at Hamming distance $\geq t$ on an m-set Σ with m distinct elements. We remind the reader that such a code is also called $(t-1)$-error detecting code or $(t-1)/2$-error correcting code because such a code is capable of detecting up to $t-1$ errors and correct up to $(t-1)/2$ errors in each transmitted code word.

5) $PBIB(b,v,r,k,\lambda_1,\lambda_2)$ denotes a partially balanced incomplete block design with b blocks each of size k, v treatments with r replication of each, and association indices λ_1 and λ_2.

6) SR-Graph (A) denotes the strongly regular graph with incidence matrix A.

7) Non #(n,t) denotes an arrangement of n mutually non attacking rooks on the t-dimensional $n \times n$ chess board.

8) PG(2,s) denotes a finite projective plane of order s (not necessarily Desargusian).

9) $\mathcal{E}$(2,s) denotes a finite Euclidean plane of order s.

10) BIB(b,v,r,k,λ) denotes a balanced incomplete block design with b blocks each of size k, v treatments with r replication of each, and association index λ.

11) $\mathfrak{K}$-Graph (A) denotes the complete graph with incidence matrix A.

12) DIF(v,k,λ) denotes a difference set with parameters v, k, and λ.

13) BLRL(s) denotes a balanced set of ℓ-restrictional lattice design for s treatments. Note that a 1-restrictional balanced lattice design is simply a BIB design.

14) HAD(n) denotes a symmetric normalized Hadamord matrix of order n.

Hereafter we also adopt the following two notations:

i) $A \Leftrightarrow B$ means A implies B and B implies A.

ii) $A \Rightarrow B$ means A implies B. Whether or not B implies A is undecided.

2. THE RESULT:

Theorem

(a) For any pair of positive integers n and t we have:

1) $O(n,t) \Leftrightarrow MOM(n,t+2)$

2) $O(n,t) \Leftrightarrow OA(n,t+2)$

3) $O(n,t) \Leftrightarrow Net(n,t+2)$

4) $O(n,t) \Leftrightarrow Code(n^2,t+2, t+1;n)$

5) $O(n,t) \Leftrightarrow PBIB(n^2,n(t+2),n,t+2,0,1)$

6) $O(n,t) \Leftrightarrow$ SR-Graph (A) where A is the incidence matrix associated with PBIB in 5).

7) $O(n,t) \Leftrightarrow Non\# (n^2,n^{t+2})$.

(b) If $t = n-1$ then also:

8) $O(n,n-1) \Leftrightarrow PG(2,n)$

9) $O(n,n-1) \Leftrightarrow \mathcal{S}(2,n)$

10) $O(n,n-1) \Leftrightarrow BIB(n^2+n+1, n^2+n+1, n+1, n+1,1)$

11) $O(n,n-1) \Leftrightarrow$ K-Graph (A) where A is the incidence matrix associated with BIB in 9).

12) $O(n,n-1) \Leftrightarrow DIF(n^2+n+1, n+1,1)$.

(c) If $n = p^m$ where p is a prime and m is a positive integer then also the following:

13) $O(p^m,p^m-1) \Leftrightarrow BLRL(p^m)$.

(d) If $n = 2r$ and $t = r-2$, $r \geq 3$ then the following are also true:

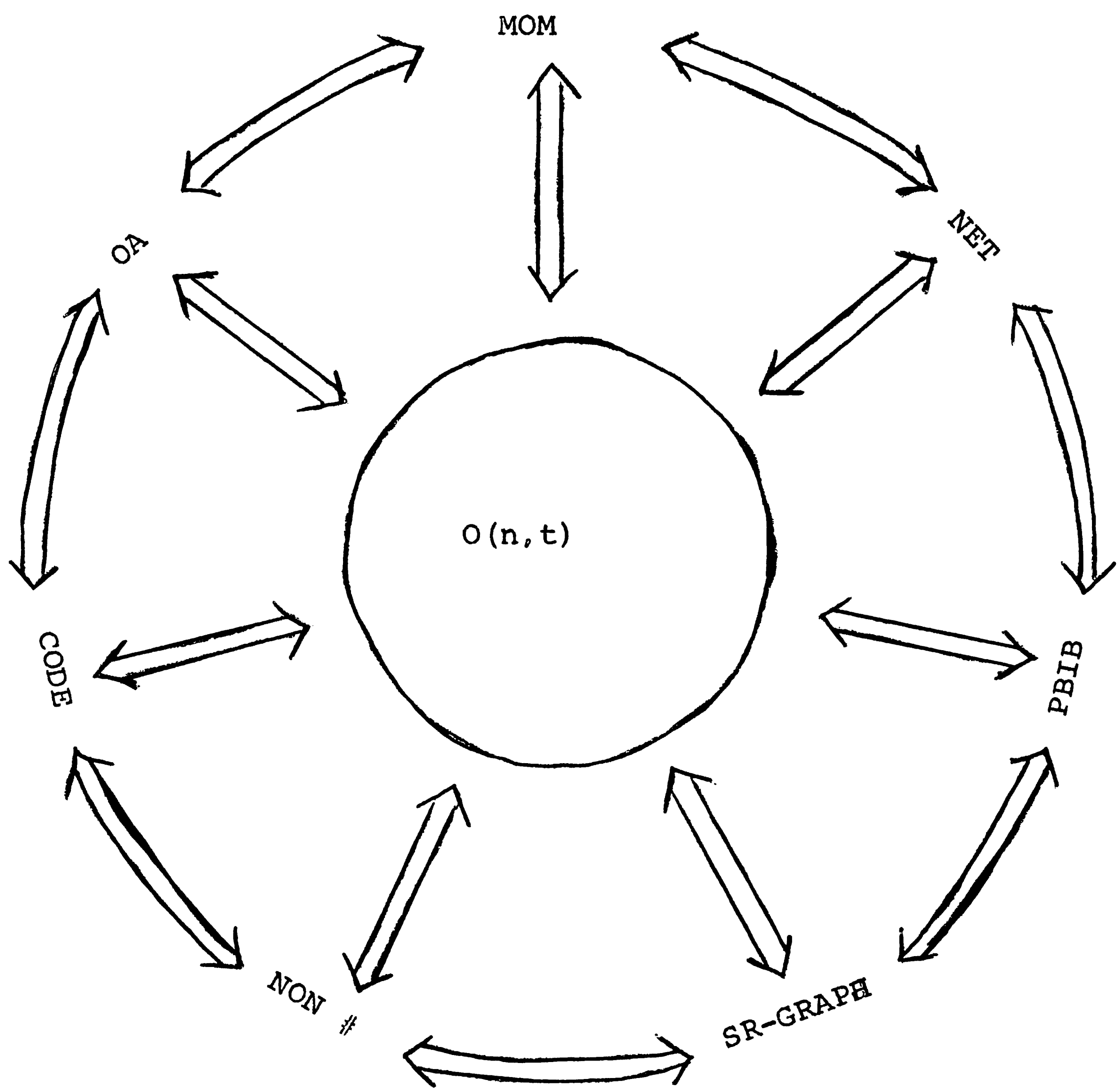
14) $O(2r,r-2) \Rightarrow HAD(4r^2)$

15) $O(2r,r-2) \Rightarrow BIB(4r^2-1,4r^2-1,2r^2-1,2r^2-1,r^2-1)$

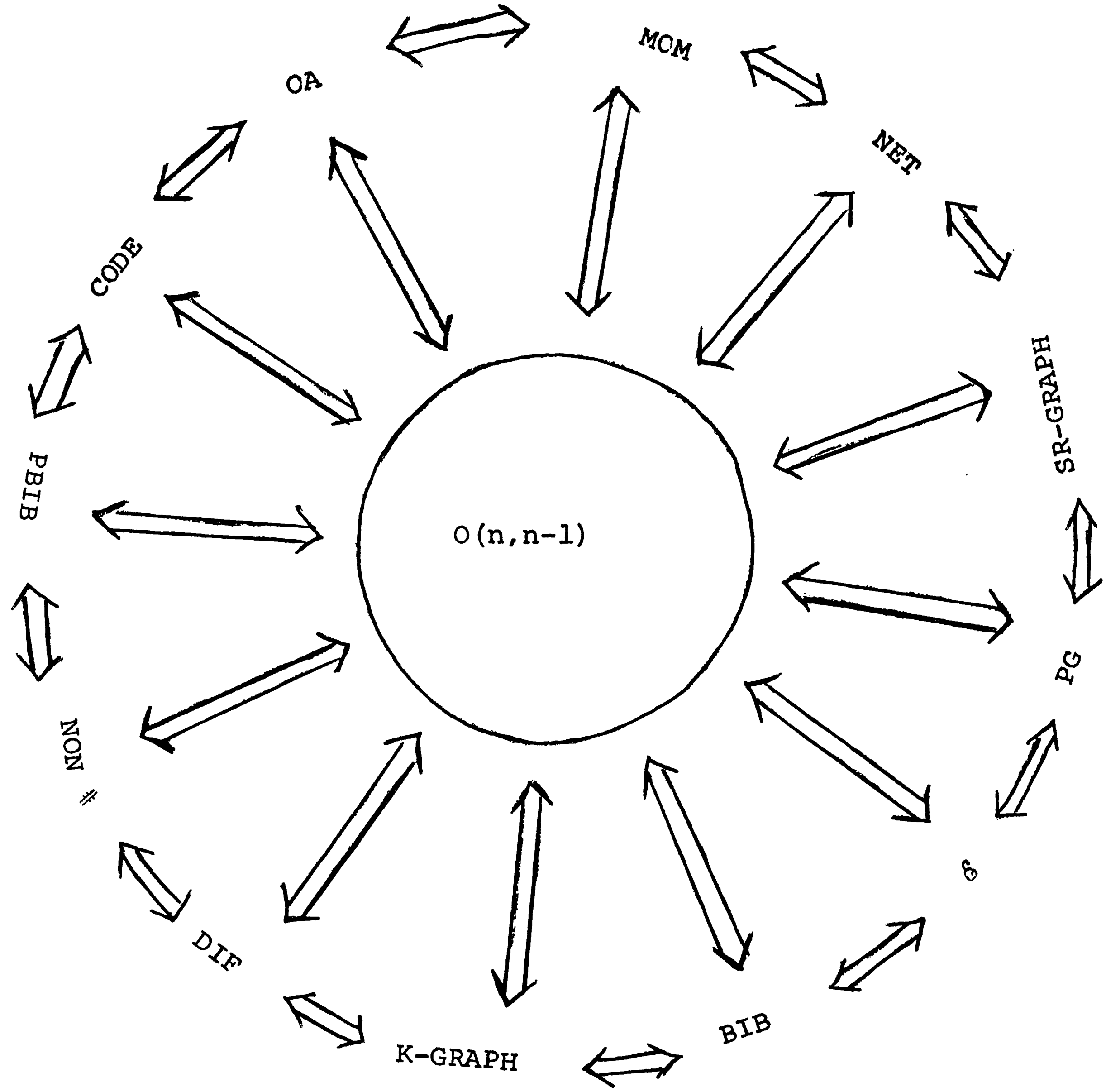
16) $O(2r,r-2) \Rightarrow Code(8r^2,4r^2,2r^2;2)$

17) $O(2r,r-2) \Rightarrow DIF(4r^2-1,2r^2-1,r^2-1)$.

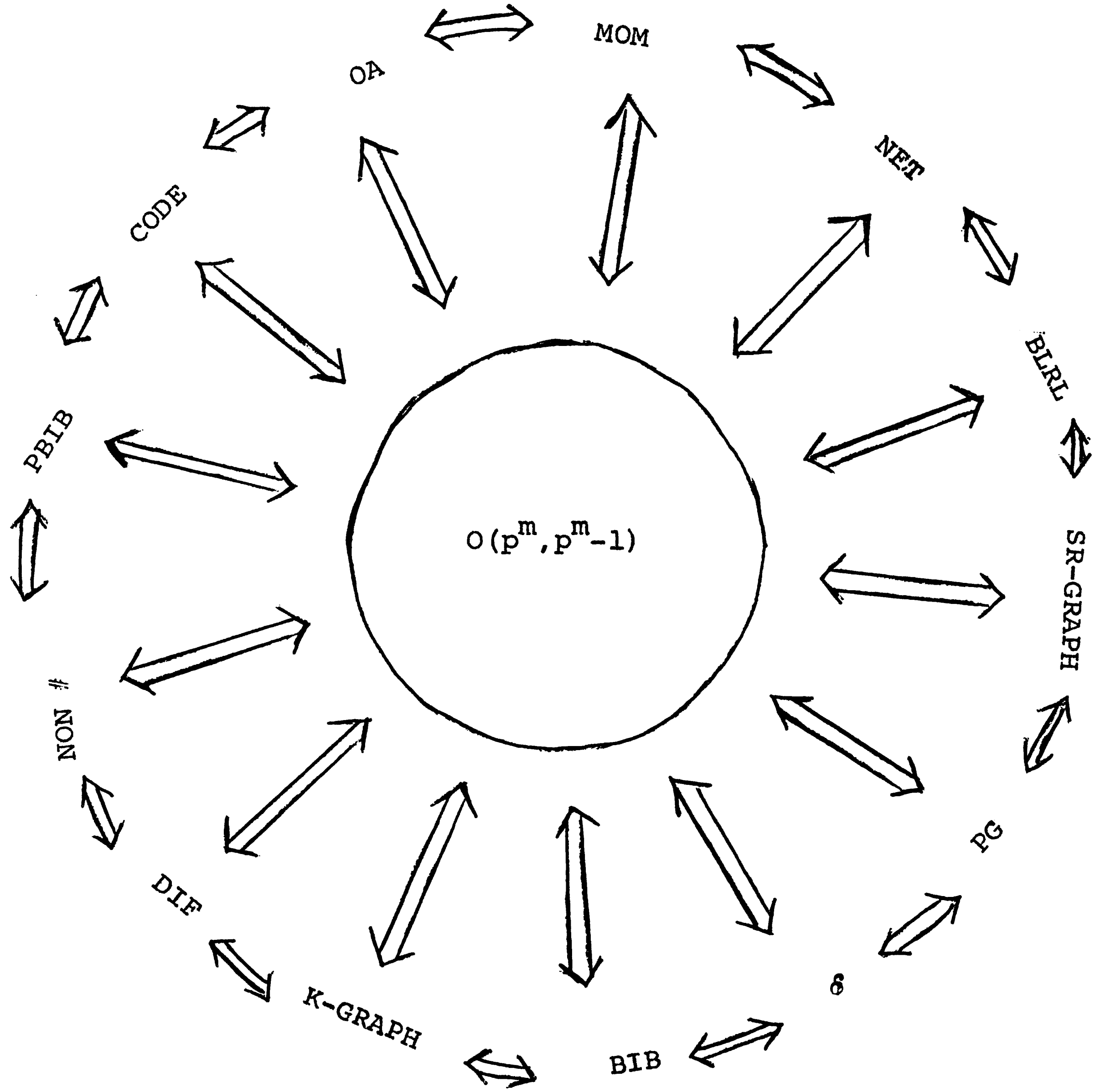
A complete schematic representation of this theorem can be demonstrated in four wheels which will be called "fundamental wheels of combinatorial mathematics". For the sake of compactness we shall omit the associated parameters with each system in these wheels except for $O(n,t)$. By knowing the values of n and t in the given $O(n,t)$ sets then the reader can easily find the associated parameters with other systems in the wheels from the proper part of above theorem.



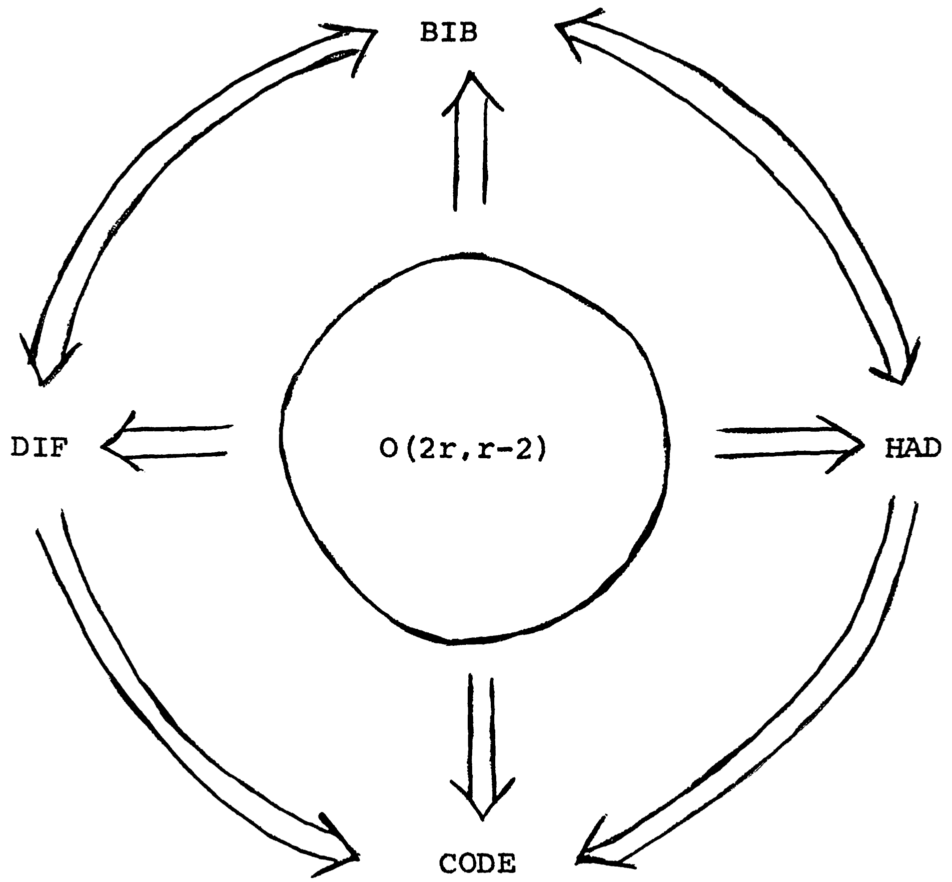
Wheel 1. For any positive integer n and t.



Wheel 2. For any positive integer n.



Wheel 3. For any prime p and positive integer m.



Wheel 4. For any positive integer $r \geq 3$.
(see also wheel 1)

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